

Topological Interference Management with User Admission Control via Riemannian Optimization

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Abstract—Topological interference management (TIM) provides a promising way to manage interference only based on the network connectivity information. Previous works on the TIM problem mainly focus on using the index coding approach and graph theory to establish conditions of network topologies to achieve the feasibility of topological interference management. In this paper, we propose a novel user admission control approach via sparse and low-rank optimization to maximize the number of admitted users for achieving the feasibility of topological interference management. To assist efficient algorithms design for the formulated rank-constrained (i.e., degrees-of-freedom (DoF) allocation) ℓ_0 -norm maximization (i.e., user capacity maximization) problem, we propose a regularized smoothed ℓ_1 -norm minimization approach to induce sparsity pattern, thereby guiding the user selection. We further develop a Riemannian trust-region algorithm to solve the resulting rank-constrained smooth optimization problem via exploiting the quotient manifold of fixed-rank matrices. Simulation results demonstrate the effectiveness and near-optimal performance of the proposed Riemannian algorithm to maximize the number of admitted users for topological interference management.

Index Terms—Topological interference management, user admission control, sparse and low-rank modeling, Riemannian optimization, quotient manifold.

I. INTRODUCTION

The popularization of innovative applications and new services, such as Internet of Things (IoT) and wearable devices [1], is driving the era of wireless big data [2], thereby revolutionizing the segments of the society. In particular, with ultra-low latency and ultra-reliable requirements, Tactile Internet [3] enables a new paradigm shift from content-delivery to skill-set delivery networks. Network densification [4], [5], supported by the advanced wireless technologies (e.g., massive MIMO [6], Cloud-RAN [7], [8], and small cells [9], [10]), becomes the key enabling technology to accommodate the exponential mobile data traffic growth, as well as provide ubiquitous connectivity for massive devices. However, by adding more radio access points per volume, interference becomes the bottleneck to harness the benefits of network densification. Although the recent development of interference alignment [11] and interference coordination [12] have been shown to be effective in the interference-limited communication scenarios, the significant signaling overhead of obtaining the global channel state information (CSI) limits applicability to dense wireless networks [13].

To reduce the CSI acquisition overhead and make it scalable in dense wireless networks, topological interference management (TIM) approach was proposed in [13] to manage interference only based on the network connectivity information. However, establishing the feasibility of topological interference management is a challenging task. In the slow fading scenario, i.e., channels stay constant during transmission, the TIM problem turns out to be equivalent to the index coding problem [14], which is, however, NP-hard in general and only some special cases have been solved [15], [13]. Furthermore, the topological interference management with transmitters cooperation and multiple transmitter antennas were investigated in [16] and [17], respectively. In the fast fading scenario, the graph theory and matroids theory were adopted to find the conditions of network topologies to achieve a certain amount of DoF allocation [18], [19]. A low-rank matrix completion approach with Riemannian algorithms has recently been proposed in [20] to find the minimum channel uses to achieve feasibility for any network topology.

In contrast, in this paper, we propose a different viewpoint: given any network topology and DoF allocation, we aim at finding the maximum number of admitted users to achieve the feasibility of topological interference management. We call this problem as *user admission control* in topological interference management. User admission control is critical in wireless communication networks (i.e., cognitive radio access networks [21], heterogeneous networks [22] and Cloud-RAN [23]) when quality-of-services (QoS) requirements are unsatisfied or the channel conditions are unfavorable [24]. Although the user admission control problems are normally non-convex mixed combinatorial optimization problems, a large body of recent work has demonstrated the effectiveness of convex relaxation for solving such problems [21], [22], [23], [24] based on the sum-of-infeasibilities in optimization theory [25]. This is achieved by relaxing the original non-convex ℓ_0 -norm minimization problem for user admission control to the convex ℓ_1 -norm minimization problem [25], [26].

Unfortunately, the user admission control problem in topological interference management turns out to be highly intractable, which needs to optimize over continuous and combinatorial variables. To address the intractability, in this paper, we propose a sparse and low-rank modeling framework to compute the proposed solutions within polynomial time. In this model, sparsity of the diagonal entries of the matrix (i.e., the number of non-zero entries) represents the number of the admitted users. The fixed low-rank constraint indicates the DoF allocation [20]. However, the unique challenges arise in the proposed sparse and low-rank optimization model

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including the non-convex fixed-rank constraint and user capacity maximization objective function, i.e., ℓ_0 -norm objective *maximization*. Novel algorithms thus need to be developed.

A. Related Works

1) *User Admission Control*: In dense wireless networks, user admission control is critical to maximize the user capacity while satisfying the QoS requirements for all the admitted users. To address the NP-hardness of the mixed combinatorial optimization problem, sparse optimization (e.g., ℓ_0 -norm minimization) approach, supported by the efficient algorithms (e.g., ℓ_1 -norm convex relaxation [22], [21] and the iterative reweighted ℓ_2 -algorithm [23]), provided an efficient way to find high quality solutions. However, convex relaxation approach is inapplicable in our sparse and low-rank optimization problem due to the ℓ_0 -norm maximization as the objective. For the ℓ_1 -norm relaxation approach, it yields a ℓ_1 -norm maximization problem, which is still non-convex. Furthermore, maximizing ℓ_1 -norm shall yield unbounded values.

2) *Low-Rank Models*: Low-rank models [27], [28] inspire enormous applications in machine learning, recommendation systems, sensor localization, etc. Due to the non-convexity of low-rank constraint or objective, many heuristic algorithms with optimality guarantees have been proposed in the last few years. In particular, convex relaxation approach using nuclear norm [29] provides a polynomial time complexity algorithm with optimality guarantees via convex geometry and conic integral geometry analysis [30].

The other popular way for low-rank optimization is based on matrix factorization, e.g., the alternating minimization [31], [28] and Riemannian optimization method [32]. In particular, the Riemannian optimization approach requires the smoothness of the objective function, while the alternating approach requires the convexity of the objective function. However, due to the non-convex and non-smooth objective function, we can not directly apply the existing matrix factorization approaches to solve the proposed sparse and low-rank optimization framework for user admission control.

Based on the above discussions, in contrast to the previous works on user admission control [21], [22], [23], [24] and low-rank optimization problems [27], [31], [28], [32], we need to address the following coupled challenges to solve the sparse and low-rank optimization for user admission control in topological interference management:

- The objective of *maximizing* the non-convex ℓ_0 -norm to maximize the user capacity, i.e., the number of admitted users;
- Non-convex fixed-rank constraint to achieve a certain amount of DoF allocation.

Therefore, unique challenges arise in the user admission control problem for topological interference management. We need to re-design the sparsity-inducing function and the efficient approach to deal with the fixed-rank constraint.

B. Contributions

In this paper, we propose a sparse and low-rank optimization framework for user admission control in topological interference management. The Riemannian trust-region algorithm is

developed to solve the proposed regularized smoothed ℓ_1 -norm sparsity inducing minimization problem, thereby guiding user selection. The main contributions are summarized as follows:

- 1) We propose a novel sparse and low-rank optimization framework to maximize the number of admitted users for achieving the feasibility of topological interference management.
- 2) To avoid unboundness in the relaxed ℓ_1 -norm maximization problem, a regularized smoothed ℓ_1 -norm is proposed to induce sparsity pattern with bounded values, thereby guiding user selection.
- 3) A Riemannian trust-region algorithm is developed to solve the resulting rank-constrained smooth optimization problem for sparsity inducing. This is achieved by exploiting the quotient manifold of fixed-rank matrices.
- 4) Simulation results demonstrate the effectiveness and near-optimal performance of the proposed Riemannian algorithm to maximize the user capacity for topological interference management.

C. Organization

The remainder of the paper is organized as follows. Section II presents the system model and problem formulation. A sparse and low-rank optimization framework for user admission control is proposed in Section III. The Riemannian optimization algorithm is developed in Section IV. The ingredients of optimization on quotient manifold are presented in Section V. Numerical results are illustrated in Section VI. Finally, conclusions and discussions are presented in Section VII.

Notations: Throughout this paper, $\|\cdot\|_p$ is the ℓ_p -norm. Boldface lower case and upper case letters represent vectors and matrices, respectively. $(\cdot)^{-1}$, $(\cdot)^T$, $(\cdot)^H$ and $\text{Tr}(\cdot)$ denote the inverse, transpose, Hermitian and trace operators, respectively. We use \mathbb{C} and \mathbb{R} to represent complex domain and real domain, respectively. $\mathbb{E}[\cdot]$ denotes the expectation of a random variable. $|\cdot|$ stands for either the size of a set or the absolute value of a scalar, depending on the context. We denote $\mathbf{A} = \text{diag}\{x_1, \dots, x_N\}$ and \mathbf{I}_N as a diagonal matrix of order N and the identity matrix of order N , respectively.

II. SYSTEM MODEL AND PROBLEM FORMULATION

In this section, we present the channel model, followed by the user admission control problem to achieve the feasibility of topological interference management.

A. Channel Model

Consider the topological interference management problem in the partially connected K -user interference channel with each node quipped with a single antenna [13], [20]. Let \mathcal{V} be the index set of the connected transceiver pairs such that the channel coefficient h_{ij} between the transmitter j and receiver i is non-zero if $(i, j) \in \mathcal{V}$, and is zero otherwise. Each transmitter i wishes to send a message W_i to its corresponding receiver i . The message W_i is encoded into a vector $\mathbf{x}_i \in \mathbb{C}^r$

of length r . Therefore, over the r channel uses, the received signal $\mathbf{y}_i \in \mathbb{C}^r$ at receiver i is given by

$$\mathbf{y}_i = h_{ii}\mathbf{x}_i + \sum_{j \in \mathcal{V}, j \neq i} h_{ij}\mathbf{x}_j + \mathbf{z}_i, \forall i = 1, \dots, K, \quad (1)$$

where $\mathbf{z}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_r)$ is the additive noise at receiver i . We consider the block fading channel, where the channel coefficients stay constant during transmission, i.e., the channel coherence time is larger than channel uses r for transmission. We assume each transmitter has an average power constraint, i.e., $\mathbb{E}[\|\mathbf{x}\|^2] \leq r\rho$ with $\rho > 0$ as the maximum average transmit power.

The rate tuple (R_1, \dots, R_K) is said to be achievable if there exists a $(2^{rR_1}, \dots, 2^{rR_K}, r)$ code scheme such that the average decoding error probability is vanishing as the code length r approaches infinity. Here, we assume that each message W_k is uniformly and independently chose over the K message sets $\mathcal{W}_k := [1 : 2^{rR_k}]$. In this paper, we choose our performance metric as the symmetric DoF [13], [16], i.e., the highest DoF achieved by all the users simultaneously,

$$d_{\text{sym}} = \limsup_{\rho \rightarrow \infty} \sup_{(R_{\text{sym}}, \dots, R_{\text{sym}}) \in \mathcal{C}} \frac{R_{\text{sym}}}{\log \rho}, \quad (2)$$

where \mathcal{C} is the capacity region defined as the set of all the achievable rate tuples. The metric of DoF gives the first-order measurement of data rates [33].

B. Topological Interference Management

In this paper, we restrict the class of the linear interference management strategies [11], [13], [20]. Specifically, each transmitter i encodes its message W_i by a linear precoding vector $\mathbf{v}_i \in \mathbb{C}^r$ over r channel uses:

$$\mathbf{x}_i = \mathbf{v}_i s_i, \quad (3)$$

where $s_i \in \mathbb{C}$ is the transmitted data symbol. Here the precoding vectors \mathbf{v}_i 's only depend on the knowledge of network topology \mathcal{V} . In this paper, we assume that the network connectivity information \mathcal{V} is available at the transmitters. Therefore, over the r channel uses, the received signal $\mathbf{y}_i \in \mathbb{C}^r$ at receiver i can be rewritten as

$$\mathbf{y}_i = h_{ii}\mathbf{v}_i s_i + \sum_{j \in \mathcal{V}, j \neq i} h_{ij}\mathbf{v}_j s_j + \mathbf{z}_i, \forall i = 1, \dots, K. \quad (4)$$

Let $\mathbf{u}_i \in \mathbb{C}^r$ be the decoding vector for each message W_i at receiver i . In the regime of asymptotically high signal-to-noise ratio (SNR), to accomplish decoding, we impose the following interference alignment condition [11], [13], [20] for the precoding and decoding vectors:

$$\mathbf{u}_k^H \mathbf{v}_k \neq 0, \forall k = 1, \dots, K, \quad (5)$$

$$\mathbf{u}_k^H \mathbf{v}_i = 0, \forall i \neq k, (i, k) \in \mathcal{V}, \quad (6)$$

where the first condition is to preserve the desired signal and the second condition is to align and cancel the interference signals. If conditions (5) and (6) are satisfied, the parallel interference-free channels can be obtained over r channel uses. Therefore, the symmetric DoF of $1/r$ is achieved for each message W_i [13]. We call this problem as *topological interference*

management [13], as only network topology information is required to establish the interference alignment conditions.

However, establishing the conditions on r , K and \mathcal{V} to achieve feasibility of the interference alignment conditions (5) and (6) is challenging. In particular, given a number of users K and channel uses r (or DoF allocation $1/r$), the index coding approach [13] and graph theory [16], [19], [18] were adopted to establish the conditions on the network topologies \mathcal{V} to achieve feasibility for the interference alignment conditions (5) and (6). The low-rank matrix completion approach [20] has recently been proposed to find the minimum number of channel uses r satisfying conditions (5) and (6), given any network topology information \mathcal{V} and the number of uses K . The feasibility conditions of antenna configuration for interference alignment in MIMO interference channel has also been extensively investigated using algebraic geometry [34], [35], [36].

In this paper, we put forth a different point of view on the feasibility conditions of topological interference management: given a number of K users with any network topology \mathcal{V} and the symmetric DoF allocation $1/r$, we present a novel user admission control approach to find the maximum number of the admitted users while satisfying the interference alignment conditions (5) and (6). Although user admission control has been extensively investigated in the scenarios of multiuser coordinated beamforming [24], cognitive radio networks [21], heterogeneous cellular networks [22] and Cloud-RAN [23], this is the first time using the principle of user admission control in the framework of topological interference management. This shall provide a systematic framework for efficient algorithms design, as well as provide numerical insights into this challenging problem of topological interference management.

III. A SPARSE AND LOW-RANK OPTIMIZATION FRAMEWORK FOR USER ADMISSION CONTROL

In this section, we present a user admission control approach to maximize the user capacity, i.e., find the maximum number of admitted users while satisfying the interference alignment conditions (5) and (6). This viewpoint is different from the previous works on finding the conditions of network topologies to achieve the feasibility of interference alignment [13], [19], [16], [18].

A. Feasibility of Interference Alignment

Given any network connectivity information \mathcal{V} for the partially connected K -user interference channel, we say that the symmetric DoF allocation $1/r$ is feasible if there exists precoding vectors $\mathbf{v}_i \in \mathbb{C}^r$ and decoding vectors $\mathbf{u}_i \in \mathbb{C}^r$ such that the interference alignment conditions (5) and (6) are satisfied. Specifically, the feasibility of topological interference management problem can be formulated as

$$\begin{aligned} \mathcal{F} : \text{find } & \{\mathbf{v}_i\}, \{\mathbf{u}_i\} \\ \text{subject to } & \mathbf{u}_i^H \mathbf{v}_i \neq 0, \forall i = 1, \dots, K, \\ & \mathbf{u}_i^H \mathbf{v}_j = 0, \forall (i, j) \in \mathcal{V}, \end{aligned} \quad (7)$$

where $\mathbf{v}_i \in \mathbb{C}^r$ and $\mathbf{u}_i \in \mathbb{C}^r$ are optimization variables.

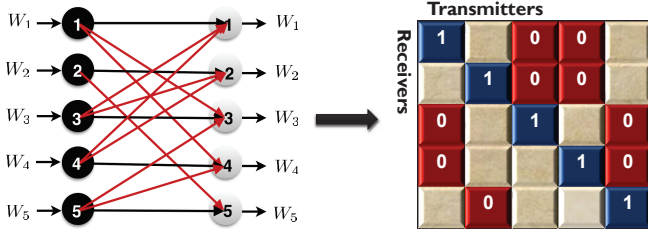


Fig. 1. (a) The topological interference alignment problem for the partially connected K -user interference channel with only the knowledge of the network connectivity information available. The interference links are marked as red while the desired links are marked as black. (b) The corresponding incomplete matrix with “0” indicating interference alignment and cancellation and “1” representing desired signal preserving.

However, the solutions to the feasibility problem (7) is unknown in general. In particular, the index coding approach [13] and the graph theory [16], [19], [18] were adopted to establish the conditions on the network topology \mathcal{V} to achieve feasibility of interference alignment. On the other hand, the low-rank matrix completion approach was proposed in [20] to find the minimum number of channel uses r to achieve interference alignment feasibility for any network topology \mathcal{V} .

In contrast, in this paper, our goal is to maximize the user capacity, i.e., the find the maximum number of admitted users while satisfying the interference alignment conditions:

$$\begin{aligned} & \text{maximize}_{\{v_i\}, \{u_i\}} |\mathcal{S}| \\ & \text{subject to } u_i^H v_i \neq 0, \forall i \in \mathcal{S}, \\ & u_i^H v_j = 0, \forall i \neq j, i, j \in \mathcal{S}, (i, j) \in \mathcal{V}, \end{aligned} \quad (8)$$

where $\mathcal{S} \subseteq \{1, \dots, K\}$ is the admitted users, $v_i \in \mathbb{C}^r$ and $u_i \in \mathbb{C}^r$. This problem is called as the *user admission control problem*. Unfortunately, it turns out to be highly intractable due to the non-convex quadratic constraints and the non-convex combinatorial objective function. To assist efficient algorithms design, in this paper, we propose a sparse and low-rank optimization for user admission control via exploiting the sparse and low-rank structures in problem (8).

B. Sparse and Low-Rank Optimization Paradigms for User Admission Control

Let $\mathbf{X} = [X_{ij}] \in \mathbb{C}^{K \times K}$ with $X_{ij} = u_i^H v_j \in \mathbb{C}$. The interference alignment conditions (5) and (6) thus can be rewritten as

$$X_{kk} \neq 0, \forall k = 1, \dots, K, \quad (9)$$

$$X_{ki} = 0, \forall i \neq k, (i, k) \in \mathcal{V}. \quad (10)$$

For other entries $X_{ki}, \forall (k, i) \notin \mathcal{V}$, they can be any values. Observing that the achievable symmetric DoF is given by

$$\text{DoF} = 1/\text{rank}(\mathbf{X}) = 1/r, \quad (11)$$

a low-rank matrix completion problem was proposed in [20] to find the minimum channel uses while satisfying the interference alignment conditions. Fig. 1 demonstrates the procedure of transforming the topological interference alignment conditions (5) and (6) into the associated incomplete matrix \mathbf{X} .

Define $\mathbf{X}(\mathcal{S}) \in \mathbb{C}^{|\mathcal{S}| \times |\mathcal{S}|}$ as the submatrix of \mathbf{X} , i.e., $\mathbf{X}(\mathcal{S}) = [X_{ij}]_{i,j \in \mathcal{S}}$. The rank of the submatrix $\mathbf{X}(\mathcal{S})$ equals r . The user admission control problem (8) can be further reformulated as follows:

$$\begin{aligned} & \text{maximize}_{\mathbf{X} \in \mathbb{C}^{K \times K}, \mathcal{S}} |\mathcal{S}| \\ & \text{subject to } \text{rank}(\mathbf{X}(\mathcal{S})) = r, \\ & X_{ii} \neq 0, \forall i \in \mathcal{S}, \\ & X_{ij} = 0, \forall i \neq j, i, j \in \mathcal{S}, (i, j) \in \mathcal{V}, \end{aligned} \quad (12)$$

where the first constraint is to preserve the symmetric DoF allocation as $1/r$. However, problem (12) is still a highly intractable mixed combinatorial optimization problem with a non-convex fixed-rank constraint and a combinatorial objective function.

To enable the capability of polynomial-time complexity algorithm design, we further reveal the sparsity structure in problem (12) for user admission control. We notice that

$$\|\text{diag}(\mathbf{X})\|_0 = |\mathcal{S}|, \quad (13)$$

where $\text{diag}(\cdot)$ extracts the diagonal of a matrix and $\|\cdot\|_0$ is the ℓ_0 -norm of a vector, i.e., the count of non-zero entries. Problem (12) can be further reformulated as the following sparse and low-rank optimization problem, i.e.,

$$\begin{aligned} \mathcal{P} : & \text{maximize}_{\mathbf{X} \in \mathbb{R}^{K \times K}} \|\text{diag}(\mathbf{X})\|_0 \\ & \text{subject to } \text{rank}(\mathbf{X}) = r, \\ & X_{ij} = 0, \forall i \neq j, (i, j) \in \mathcal{V}. \end{aligned} \quad (14)$$

Notice that we only need to consider problem \mathcal{P} in the real field without losing any performance in terms of admitted users. The reason is that the affine constrain (14) is restricted in real field and the diagonal entries of matrix \mathbf{X} can be further restricted to the real field while achieving the same value of $\|\text{diag}(\mathbf{X})\|_0$ in the complex field.

Sparse optimization has shown to be powerful for the user admission problems [24], [21], [22], [23] via ℓ_0 -norm minimization using the sum-of-infeasibilities convex relaxation heuristic in optimization theory [25, Section 11.4]. In particular, to maximize the number of admitted users is equivalent to minimize the number of violated inequalities for the quality-of-service (QoS) constraints. Although problem \mathcal{P} adopts the same philosophy of ℓ_0 -norm to count the number of admitted users (13), it reveals unique challenges due to ℓ_0 -norm maximization and non-convex fixed-rank constraint. However, compared with the original formulation (12), the sparse and low-rank optimization formulation (14) holds algorithmic advantages, which are demonstrated in the sequel via the Riemannian optimization approach [37].

C. Problem Analysis

In this subsection, we reveal the unique challenges of solving the sparse and low-rank optimization problem \mathcal{P} for user admission control in topological interference management.

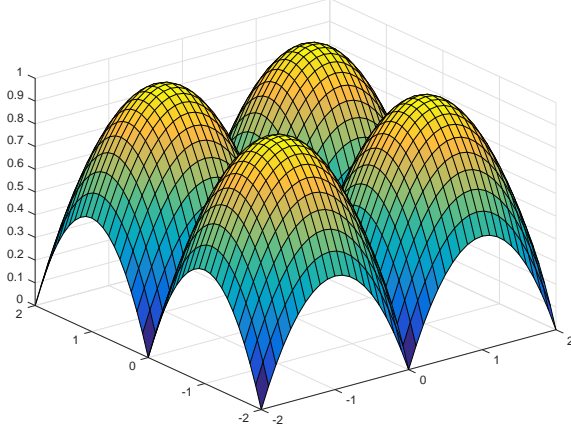


Fig. 2. The regularized sparsity inducing norm $f(\mathbf{z}) = \|\mathbf{z}\|_1 - 0.5\|\mathbf{z}\|_2^2$ with bounded values in $\mathbf{z} \in \mathbb{R}^2$.

1) *Non-convex Objective Function:* Although ℓ_1 -norm serves the convex surrogate for the non-convex ℓ_0 -norm [25], [26], it is inapplicable in problem \mathcal{P} for ℓ_0 -norm *maximization*, as it yields unbounded values. To aid efficient algorithms design, we propose a novel regularized ℓ_1 -norm to induce sparsity with bounded values. This is achieved by adding a quadratic term in the ℓ_1 -norm as follows:

$$f(\mathbf{z}) = \|\mathbf{z}\|_1 - \lambda \|\mathbf{z}\|_2^2, \quad (15)$$

where $\mathbf{z} \in \mathbb{R}^n$ and $\lambda \geq 0$ is a weighting parameter. A typical example with $f(\mathbf{z}) = \|\mathbf{z}\|_1 - 0.5\|\mathbf{z}\|_2^2$ and $\mathbf{z} \in \mathbb{R}^2$ is illustrated in Fig. 2, which upper bounds all the diagonal values by 1.

2) *Non-convex Fixed-rank Constraint:* Matrix factorization serves a powerful way to address the non-convexity of the fixed-rank matrices. One popular way is to factorize a fixed rank- r matrix \mathbf{X} (in real field) as $\mathbf{U}\mathbf{V}^T$ with $\mathbf{U} \in \mathbb{R}^{K \times r}$ and $\mathbf{V} \in \mathbb{R}^{K \times r}$, followed by alternatively optimizing over \mathbf{U} and \mathbf{V} holding the other fixed [28], [31]. However, due to the non-convex objective function in problem \mathcal{P} , the resulting optimization problem over \mathbf{U} or \mathbf{V} is still non-convex. Furthermore, such factorization is not unique as \mathbf{X} remains unchanged under the transformation of the factors

$$(\mathbf{U}, \mathbf{V}) \mapsto (\mathbf{U}\mathbf{M}^{-1}, \mathbf{V}\mathbf{M}^T), \quad (16)$$

for all non-singular matrices \mathbf{M} of size $r \times r$. As a result, the critical points of an objective function parameterized with \mathbf{U} and \mathbf{V} are *not isolated* on $\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$. This profoundly affects the performance of second-order optimization algorithms which require non degenerate critical points, which is no longer the case here. We propose to address this issue by exploiting the *quotient manifold geometry* of the set of fixed-rank matrices [38]. The resulting non-convex optimization problem is further solved by exploiting the Riemannian optimization framework which provides systematic ways to develop algorithms on quotient manifolds [37].

In summary, in this paper, we propose a new powerful approach to induce the sparsity in the solution to problem \mathcal{P} , followed by the Riemannian optimization approach via



Fig. 3. The proposed three-stage Riemannian framework for user admission control in topological interference alignment via sparse and low-rank optimization. $\mathbf{z}^* \in \mathbb{R}^K$ is the induced sparsity pattern for user selection and $\mathcal{S}^* \subseteq \{1, \dots, K\}$ is set of admitted users.

exploiting the quotient manifold geometry of fixed-rank matrices. The induced sparsity pattern guides user selection for user admission control.

IV. REGULARIZED SMOOTHED ℓ_1 -MINIMIZATION FOR SPARSE AND LOW-RANK OPTIMIZATION VIA RIEMANNIAN OPTIMIZATION

In this section, we present a Riemannian framework for sparse and low-rank optimization problem \mathcal{P} via regularized smoothed ℓ_1 -minimization by exploiting the quotient manifold geometry of fixed-rank matrices. The induced sparsity solution to problem \mathcal{P} provides guideline for user admission control, supported by a user selection procedure. In the final stage, a low-rank matrix completion approach with Riemannian optimization is adopted to design the linear topological interference management strategy. The proposed three-stage Riemannian framework for user admission control in topological interference management is presented in Fig. 3.

A. Stage One: Regularized Smoothed ℓ_1 -Minimization for Sparsity Inducing

In order to make problem \mathcal{P} (14) numerically tractable, we relax the non-convex ℓ_0 -norm objective function to its convex surrogate ℓ_1 -norm, resulting in the following optimization problem:

$$\begin{aligned} & \underset{\mathbf{X} \in \mathbb{R}^{K \times K}}{\text{maximize}} && \|\text{diag}(\mathbf{X})\|_1 \\ & \text{subject to} && \text{rank}(\mathbf{X}) = r, \\ & && X_{ij} = 0, \forall i \neq j, (i, j) \in \mathcal{V}. \end{aligned} \quad (17)$$

Although the ℓ_1 -norm is tractable, it is *unbounded* from above due to ℓ_1 -norm *maximization*, which makes problem (17) ill-posed. Note that maximizing a convex ℓ_1 -norm is still non-convex.

To circumvent the unboundness issue, we add the quadratic term $-\lambda \|\text{diag}(\mathbf{X})\|_2^2$ to the objective function in problem (17), where $\lambda \geq 0$ is a weighting parameter that bounds the overall objective function from above leading to the formulation

$$\begin{aligned} & \underset{\mathbf{X} \in \mathbb{R}^{K \times K}}{\text{maximize}} && \|\text{diag}(\mathbf{X})\|_1 - \lambda \|\text{diag}(\mathbf{X})\|_2^2 \\ & \text{subject to} && \text{rank}(\mathbf{X}) = r, \\ & && X_{ij} = 0, \forall i \neq j, (i, j) \in \mathcal{V}. \end{aligned} \quad (18)$$

For example, if $\lambda = 0.5$, then the diagonal values of \mathbf{X} are upper bounded by 1. It should be emphasized that the role of λ in (18) is to upper bound the objective function and it does not affect the sparsity pattern that is expected from (17). This is further confirmed in Section IV-D via simulations. Additionally, if \mathbf{X}^* is the solution to (14), then $\alpha \mathbf{X}^*$ is also a solution of (14) for all non-zero scalar α . Equivalently, there

exists continuum of solutions, which is effectively resolved by the objective function in (18).

Although problem (18) is still non-convex due to the non-convex objective (i.e., maximizing a convex function) and non-convex fixed-rank constraint, it has the algorithmic advantage that it can be solved efficiently (i.e., numerically) in the framework of Riemannian optimization [37].

1) *Riemannian Optimization for Fixed-Rank Optimization:* In this subsection, we propose a Riemannian optimization algorithm to solve the non-convex optimization problem (18), which is equivalent to

$$\begin{aligned} & \underset{\mathbf{X} \in \mathbb{R}^{K \times K}}{\text{minimize}} && -\|\text{diag}(\mathbf{X})\|_1 + \lambda \|\text{diag}(\mathbf{X})\|_2^2 \\ & \text{subject to} && \text{rank}(\mathbf{X}) = r, \\ & && X_{ij} = 0, \forall i \neq j, (i, j) \in \mathcal{V}. \end{aligned} \quad (19)$$

However, the intersection of rank constraint and the affine constraint is challenging to characterize. We, therefore, propose to solve problem (19) via a *regularized* version as follows:

$$\begin{aligned} \mathcal{P}_{\text{RS}} : \underset{\mathbf{X} \in \mathbb{C}^{K \times K}}{\text{minimize}} & \quad \underbrace{\frac{1}{2} \sum_{(i,j) \in \mathcal{V}} X_{ij}^2}_{\text{network topology}} + \underbrace{\rho \sum_{i=1}^K (\lambda X_{ii}^2 - (X_{ii}^2 + \epsilon^2)^{1/2})}_{\text{admission}} \\ & \text{subject to} \quad \text{rank}(\mathbf{X}) = r, \end{aligned} \quad (20)$$

where $\rho \geq 0$ is the regularization parameter and ϵ is the parameter that approximates $|X_{ii}|$ with the smooth term $(X_{ii}^2 + \epsilon^2)^{1/2}$ that makes the objective function *differentiable*. A very small ϵ leads to ill-conditioning of the objective function in (20). Since we intend to obtain the sparsity pattern of the optimal \mathbf{X} , we set ϵ to a high value, e.g., 0.01, to make problem (20) well conditioned. Problem \mathcal{P}_{RS} is an optimization problem over the set of fixed-rank matrices and can be solved via a Riemannian trust-region algorithm [37].

B. Stage Two: Finding Sparsity Pattern for User Admission Control

Let \mathbf{X}^* be the solution to the regularized smoothed ℓ_1 -minimization problem \mathcal{P}_{RS} . We order the diagonal entries of matrix \mathbf{X}^* , i.e., the vector $\mathbf{z}^* = \text{diag}(\mathbf{X}^*) \in \mathbb{R}^K$, in the descending order: $|z_{\pi_1}| \geq |z_{\pi_2}| \geq \dots \geq |z_{\pi_K}|$. The user with larger coefficients z_i has a higher priority to be admitted. We adopt the bi-section search procedure to find the maximum number of admitted users. Specifically, let N_0 be the maximum number of users that can be admitted while satisfying the interference alignment conditions. To determine the value of N_0 , a sequence of the following size-reduced topological interference management feasibility problem needs to be solved,

$$\begin{aligned} \mathcal{F}(\mathcal{S}^{[m]}) : \text{find} & \quad \mathbf{X}(\mathcal{S}^{[m]}) \\ & \text{subject to} \quad \text{rank}(\mathbf{X}(\mathcal{S}^{[m]})) = r, \\ & \quad X_{ii} = 1, \forall i \in \mathcal{S}^{[m]}, \\ & \quad X_{ij} = 0, \forall i \neq j, i, j \in \mathcal{S}^{[m]}, (i, j) \in \mathcal{V}, \end{aligned} \quad (21)$$

where $\mathcal{S}^{[m]} = \{\pi_1, \dots, \pi_m\}$.

To check the feasibility, we rewrite problem (21) as follows:

$$\begin{aligned} & \text{minimize} \quad \|\mathcal{P}_{\Omega}(\mathbf{X}(\mathcal{S}^{[m]})) - \mathbf{I}_{|\mathcal{S}^{[m]}|}\|_F^2 \\ & \text{subject to} \quad \text{rank}(\mathbf{X}(\mathcal{S}^{[m]})) = r, \end{aligned} \quad (22)$$

where $\Omega = \{(i, j) | i, j \in \mathcal{S}^{[m]}, (i, j) \in \mathcal{V}\}$ and $\mathcal{P}_{\Omega}(\mathbf{Y}) : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is the orthogonal projection operator onto the subspace of matrices which vanish outside Ω such that the (i, j) -th component of $\mathcal{P}_{\Omega}(\mathbf{Y})$ equals to Y_{ij} if $(i, j) \in \Omega$ and zero otherwise. If the objective value approaches to zero, we say that the set of users $\mathcal{S}^{[m]}$ can be admitted. Problem (22) can be solved by Riemannian trust-region algorithms [39] via Manopt [40]. Note that, theoretically, the Riemannian algorithm can only guarantee convergence to a first-order critical point, but empirically, we observe convergence to critical points that are local minima.

C. Stage Three: Low-Rank Matrix Completion for Topological Interference Management

Let $\mathcal{S}^* = \{\pi_1, \dots, \pi_{N_0}\}$ be the admitted users. We need to solve the following sized-reduced rank-constrained *matrix completion* problem:

$$\begin{aligned} \mathcal{P}_{\text{LRMC}}(\mathcal{S}^*) : \text{minimize} & \quad \|\mathcal{P}_{\Omega}(\mathbf{X}(\mathcal{S}^*)) - \mathbf{I}_{|\mathcal{S}^*|}\|_F^2 \\ & \text{subject to} \quad \text{rank}(\mathbf{X}(\mathcal{S}^*)) = r, \end{aligned} \quad (23)$$

to find the precoding vectors \mathbf{v}_i 's and decoding vectors \mathbf{u}_i 's for the admitted users in \mathcal{S}^* .

Therefore, the proposed three-stage Riemannian optimization based user admission control algorithm is presented in Algorithm 1.

Algorithm 1: User Admission Control for Topological Interference Management via Riemannian Optimization

Step 0: Solve the sparse inducing optimization problem \mathcal{P}_{RS} (20) using the Riemannian trust-region algorithm in Section V. Obtain the solution \mathbf{X}^* and sort the diagonal entries in the descending order: $|z_{\pi_1}| \geq \dots \geq |z_{\pi_K}|$, **go to Step 1.**

Step 1: Initialize $N_{\text{low}} = 0$, $N_{\text{up}} = K$, $i = 0$.

Step 2: Repeat

- 1) Set $i \leftarrow \left\lfloor \frac{N_{\text{low}} + N_{\text{up}}}{2} \right\rfloor$.
- 2) Solve problem $\mathcal{F}(\mathcal{S}^{[i]})$ (21) via (22) using the Riemannian trust-region algorithm in Section V: if it is feasible, set $N_{\text{low}} = i$; otherwise, set $N_{\text{up}} = i$.

Step 3: Until $N_{\text{up}} - N_{\text{low}} = 1$, obtain $N_0 = N_{\text{up}}$ and obtain the admitted users set $\mathcal{S}^* = \{\pi_1, \dots, \pi_{N_0}\}$.

Step 4: Solve problem $\mathcal{P}_{\text{LRMC}}(\mathcal{S}^*)$ (23) to obtain the precoding and decoding vectors for the admitted users.

End

D. The Framework of Fixed-Rank Riemannian Manifold Optimization

The optimization problems (20), (22), and (23) are *least-square* optimization problems with fixed rank constraint. A

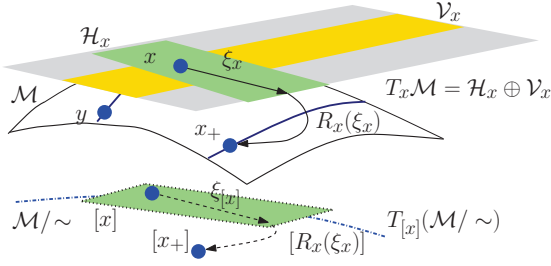


Fig. 4. Optimization on a quotient manifold. The dotted lines represent abstract objects and the solid lines are their matrix representations. The points x and y in the total (computational) space \mathcal{M} belong to the same equivalence class (shown in solid blue color) and they represent a single point $[x] := \{y \in \mathcal{M} : y \sim x\}$ in the quotient space \mathcal{M}/\sim . An algorithm by necessity is implemented in the computation space, but conceptually, the search is on the quotient manifold. Given a search direction ξ_x at x , the updated point on \mathcal{M} is given by the retraction mapping R_x .

rank- r matrix $\mathbf{X} \in \mathbb{R}^{K \times K}$ is parameterized as $\mathbf{X} = \mathbf{U}\mathbf{V}^T$, where $\mathbf{U} \in \mathbb{R}^{K \times r}$ and $\mathbf{V} \in \mathbb{R}^{K \times r}$ are full column-rank matrices. Such a factorization, however, is not unique as \mathbf{X} remains unchanged under the transformation of the factors

$$(\mathbf{U}, \mathbf{V}) \mapsto (\mathbf{U}\mathbf{M}^{-1}, \mathbf{V}\mathbf{M}^T), \quad (24)$$

for all non-singular matrices $\mathbf{M} \in \text{GL}(r)$, the set of $r \times r$ non-singular matrices. Equivalently, $\mathbf{X} = \mathbf{U}\mathbf{V}^T = \mathbf{U}\mathbf{M}^{-1}(\mathbf{V}\mathbf{M}^T)^T$ for all non-singular matrices \mathbf{M} . As a result, the critical points of an objective function parameterized with \mathbf{U} and \mathbf{V} are *not isolated* on $\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$.

The classical remedy to remove this indeterminacy requires further (triangular-like) structure in the factors \mathbf{U} and \mathbf{V} . For example, LU decomposition is a way forward. In contrast, we encode the invariance map (24) in an abstract search space by optimizing directly over a set of equivalence classes

$$[(\mathbf{U}, \mathbf{V})] := \{(\mathbf{U}\mathbf{M}^{-1}, \mathbf{V}\mathbf{M}^T) : \mathbf{M} \in \text{GL}(r)\}. \quad (25)$$

The set of equivalence classes is termed as the *quotient space* and is denoted by

$$\mathcal{M}_r := \mathcal{M}/\text{GL}(r), \quad (26)$$

where the total space \mathcal{M} is the product space $\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$.

Consequently, if an element $x \in \mathcal{M}$ has the matrix characterization (\mathbf{U}, \mathbf{V}) , then (20), (22), and (23) are of the form

$$\underset{[x] \in \mathcal{M}_r}{\text{minimize}} \quad f([x]), \quad (27)$$

where $[x] = [(\mathbf{U}, \mathbf{V})]$ is defined in (25) and $f : \mathcal{M} \rightarrow \mathbb{R} : x \mapsto f(x)$ is a *smooth* function on \mathcal{M} , but now induced (with slight abuse of notation) on the quotient space \mathcal{M}_r (26).

The quotient space \mathcal{M}_r has the structure of a smooth *Riemannian* quotient manifold of \mathcal{M} by $\text{GL}(r)$ [38]. The Riemannian structure conceptually transforms a rank-constrained optimization problem into an *unconstrained* optimization problem over the non-linear manifold \mathcal{M}_r . Additionally, it allows to compute objects like gradient (of an objective function) and develop a Riemannian trust-region algorithm on \mathcal{M}_r that uses second-order information for faster convergence [37].

V. OPTIMIZATION ON QUOTIENT MANIFOLD

Consider an equivalence relation \sim in the *total* (computational) space \mathcal{M} . The quotient manifold \mathcal{M}/\sim generated by this equivalence property consists of elements that are *equivalence classes* of the form $[x] = \{y \in \mathcal{M} : y \sim x\}$. Equivalently, if $[x]$ is an element in \mathcal{M}/\sim , then its matrix representation in \mathcal{M} is x . In the context of rank constraint, \mathcal{M}/\sim is identified with \mathcal{M}_r , i.e., the fixed-rank manifold. Fig. 4 shows a schematic viewpoint of optimization on a quotient manifold. Particularly, we need the notion of “linearization” of the search space, “search” direction, and a way “move” on a manifold. Below we show the concrete development of these objects that allow to do develop a second-order trust-region algorithm on manifolds. The concrete manifold-related ingredients are shown in Table I, which are based on the developments in [41].

Since the manifold \mathcal{M}/\sim is an abstract space, the elements of its tangent space $T_{[x]}(\mathcal{M}/\sim)$ at $[x]$ also call for a matrix representation in the tangent space $T_x\mathcal{M}$ that respects the equivalence relation \sim . Equivalently, the matrix representation of $T_{[x]}(\mathcal{M}/\sim)$ should be restricted to the directions in the tangent space $T_x\mathcal{M}$ on the total space \mathcal{M} at x that do not induce a displacement along the equivalence class $[x]$. This is realized by decomposing $T_x\mathcal{M}$ into complementary subspaces, the *vertical* and *horizontal* subspaces such that $\mathcal{V}_x \oplus \mathcal{H}_x = T_x\mathcal{M}$. The vertical space \mathcal{V}_x is the tangent space of the equivalence class $[x]$. On the other hand, the horizontal space \mathcal{H}_x , which is any complementary subspace to \mathcal{V}_x in $T_x\mathcal{M}$, provides a valid matrix representation of the abstract tangent space $T_{[x]}(\mathcal{M}/\sim)$ [37, Section 3.5.8]. An abstract tangent vector $\xi_{[x]} \in T_{[x]}(\mathcal{M}/\sim)$ at $[x]$ has a unique element in the horizontal space $\xi_x \in \mathcal{H}_x$ that is called its *horizontal lift*. Our specific choice of the horizontal space is the subspace of $T_x\mathcal{M}$ that is the *orthogonal complement* of \mathcal{V}_x in the sense of a Riemannian metric (an inner product).

A Riemannian metric or an inner product $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \rightarrow \mathbb{R}$ at $x \in \mathcal{M}$ in the total space defines a Riemannian metric $g_{[x]} : T_{[x]}(\mathcal{M}/\sim) \times T_{[x]}(\mathcal{M}/\sim) \rightarrow \mathbb{R}$, i.e.,

$$g_{[x]}(\xi_{[x]}, \eta_{[x]}) := g_x(\xi_x, \eta_x), \quad (28)$$

on the quotient manifold \mathcal{M}/\sim , provided that the expression $g_x(\xi_x, \eta_x)$ does not depend on a specific representation along the equivalence class $[x]$. Here $\xi_{[x]}$ and $\eta_{[x]}$ are tangent vectors in $T_{[x]}(\mathcal{M}/\sim)$, and ξ_x, η_x are their horizontal lifts in \mathcal{H}_x at x . Equivalently, if y is another element that belongs to $[x]$ and ξ_y and η_y are the horizontal lifts of $\xi_{[x]}$ and $\eta_{[x]}$ at y , then the metric in (28) obeys the equality $g_x(\xi_x, \eta_x) = g_y(\xi_y, \eta_y)$. Such a metric is then said to be *invariant* to the equivalence relation \sim .

In the context of fixed-rank matrices, there exist metrics which are invariant. A particular invariant Riemannian metric on the total space \mathcal{M} that takes into account the symmetry (24) imposed by the factorization model and that is well suited to a least-squares objective function [41] is

$$g_x(\xi_x, \eta_x) = \text{Tr}((\mathbf{V}^T \mathbf{V}) \xi_U^T \eta_U) + \text{Tr}((\mathbf{U}^T \mathbf{U}) \xi_V^T \eta_V), \quad (29)$$

where $x = (\mathbf{U}, \mathbf{V})$ and $\xi_x, \eta_x \in T_x\mathcal{M}$. It should be noted that the tangent space $T_x\mathcal{M}$ has the matrix characterization

TABLE I
MANIFOLD-RELATED INGREDIENTS

	$\mathbf{X} = \mathbf{U}\mathbf{V}^T$
Matrix representation	$x = (\mathbf{U}, \mathbf{V})$
Total space \mathcal{M}	$\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$
Group action	$(\mathbf{U}\mathbf{M}^{-1}, \mathbf{V}\mathbf{M}^T), \mathbf{M} \in \text{GL}(r)$
Quotient space \mathcal{M}/\sim	$\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r} / \text{GL}(r)$
Vectors in the ambient space	$(\mathbf{Z}_U, \mathbf{Z}_V) \in \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$
Matrix representation of a tangent vector ξ_x in $T_x\mathcal{M}$	$(\xi_U, \xi_V) \in \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$
Metric $g_x(\xi_x, \eta_x)$ for any $\xi_x, \eta_x \in T_x\mathcal{M}$	$\text{Tr}((\mathbf{V}^T \mathbf{V}) \xi_U^T \eta_U) + \text{Tr}((\mathbf{U}^T \mathbf{U}) \xi_V^T \eta_V)$
Vertical tangent vectors in \mathcal{V}_x	$\{(-\mathbf{U}\mathbf{\Lambda}, \mathbf{V}\mathbf{\Lambda}^T) : \mathbf{\Lambda} \in \mathbb{R}^{r \times r}\}$
Horizontal tangent vectors in \mathcal{H}_x	$\{(\zeta_U, \zeta_V) \in \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r} : \mathbf{U}^T \zeta_U \mathbf{V}^T \mathbf{V} = \mathbf{U}^T \mathbf{U} \zeta_V^T \mathbf{V}\}$
Projection of a tangent vector $\eta_x \in T_x\mathcal{M}$ on the horizontal space \mathcal{H}_x	$\Pi_x(\eta_x) = (\eta_U + \mathbf{U}\mathbf{\Lambda}, \eta_V - \mathbf{V}\mathbf{\Lambda}^T)$, where $\mathbf{\Lambda} = 0.5(\eta_V^T \mathbf{V}(\mathbf{V}^T \mathbf{V})^{-1} - (\mathbf{U}^T \mathbf{U})^{-1} \mathbf{U}^T \eta_U)$.
Retraction of a horizontal vector ξ_x onto the manifold	$R_x(\xi_x) = (\mathbf{U} + \xi_U, \mathbf{V} + \xi_V)$
Matrix representation of the Riemannian gradient $\text{grad}_x f$	$(\frac{\partial f}{\partial \mathbf{U}}(\mathbf{V}^T \mathbf{V})^{-1}, \frac{\partial f}{\partial \mathbf{V}}(\mathbf{U}^T \mathbf{U})^{-1})$, where $\partial f / \partial \mathbf{U}$ and $\partial f / \partial \mathbf{V}$ are the partial derivatives of f with respect to \mathbf{U} and \mathbf{V} , respectively.
Matrix representation of the Riemannian Hessian $\text{Hess}_x f[\xi_x]$ along a horizontal vector ξ_x	$\Pi_x(\nabla_{\xi_x} \text{grad}_x f)$, where $\text{grad}_x f$ has the representation shown above. The matrix representation of the Riemannian connection $\nabla_{\xi_x} \eta_x$ is shown in (34). Finally, the projection operator Π_x is defined in (31).

$\mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$, i.e., η_x (and similarly ξ_x) has the matrix representation $(\eta_U, \eta_V) \in \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$.

To show that (29) is invariant to the transformation (24), we assume that another element $y \in [x]$ has matrix representation $(\mathbf{U}\mathbf{M}^{-1}, \mathbf{V}\mathbf{M})$ for a non singular square matrix \mathbf{M} . Similarly, we assume that the tangent vector η_y (similarly ξ_y) has matrix representation $(\eta_{\mathbf{U}\mathbf{M}^{-1}}, \eta_{\mathbf{V}\mathbf{M}^T}) \in \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r}$. If η_x and η_y (similarly for ξ_x and ξ_y) are the *horizontal lifts* of $\eta_{[x]}$ at x and y , respectively. Then, we have $\eta_{\mathbf{U}\mathbf{M}^{-1}} = \eta_U \mathbf{M}^{-1}$ and $\eta_{\mathbf{V}\mathbf{M}^T} = \eta_V \mathbf{M}^T$ [37, Example 3.5.4]. Similarly for ξ_y . A few computations then show that $g_x(\xi_x, \eta_x) = g_y(\xi_y, \eta_y)$, which implies that the metric (29) is invariant to the transformation (24) along the equivalence class $[x]$. This implies that we have a unique metric on the quotient space \mathcal{M}/\sim .

Motivation for the metric (29) comes from the fact that it is induced from a *block diagonal* approximation of the Hessian of a simpler cost function $\|\mathbf{U}\mathbf{V}^T - \mathbf{I}\|_F^2$, which is strictly convex in \mathbf{U} and \mathbf{V} individually. This block diagonal approximation ensures that the cost of computing (29) depends linearly on K and the metric is well suited for least-squares problems. Similar ideas have also been exploited in [20], [42], [43] which show robust performance of Riemannian algorithms for various least-squares problems.

Once the metric (29) is defined on \mathcal{M} , the development of

the geometric objects required for second-order optimization follow [37], [41]. The matrix characterizations of the tangent space $T_x\mathcal{M}$, vertical space \mathcal{V}_x , and horizontal space \mathcal{H}_x are straightforward with the expressions:

$$\begin{aligned} T_x\mathcal{M} &= \mathbb{R}^{K \times r} \times \mathbb{R}^{K \times r} \\ \mathcal{V}_x &= \{(-\mathbf{U}\mathbf{\Lambda}, \mathbf{V}\mathbf{\Lambda}^T) : \mathbf{\Lambda} \in \mathbb{R}^{r \times r}\} \\ \mathcal{H}_x &= \{(\zeta_U, \zeta_V) : \mathbf{U}^T \zeta_U \mathbf{V}^T \mathbf{V} = \mathbf{U}^T \mathbf{U} \zeta_V^T \mathbf{V}, \\ &\quad \zeta_U, \zeta_V \in \mathbb{R}^{K \times r}\}. \end{aligned} \quad (30)$$

Apart from the characterization of the horizontal space, we need a linear mapping $\Pi_x : T_x\mathcal{M} \mapsto \mathcal{H}_x$ that projects vectors from the tangent space onto the horizontal space. Projecting an element $\eta_x \in T_x\mathcal{M}$ onto the horizontal space is accomplished with the operator

$$\Pi_x(\eta_x) = (\eta_U + \mathbf{U}\mathbf{\Lambda}, \eta_V - \mathbf{V}\mathbf{\Lambda}^T), \quad (31)$$

where $\mathbf{\Lambda} \in \mathbb{R}^{r \times r}$ is uniquely obtained by ensuring that $\Pi_x(\eta_x)$ belongs to the horizontal space characterized in (30). Finally, the expression of $\mathbf{\Lambda}$ is

$$\begin{aligned} \mathbf{U}^T(\eta_U + \mathbf{U}\mathbf{\Lambda})\mathbf{V}^T \mathbf{V} &= \mathbf{U}^T \mathbf{U}(\eta_V - \mathbf{V}\mathbf{\Lambda}^T)^T \mathbf{V} \\ \Rightarrow \mathbf{\Lambda} &= 0.5(\eta_V^T \mathbf{V}(\mathbf{V}^T \mathbf{V})^{-1} - (\mathbf{U}^T \mathbf{U})^{-1} \mathbf{U}^T \eta_U). \end{aligned}$$

A. Gradient and Hessian Computations

The choice of the metric (29) and of the horizontal space (as the orthogonal complement of \mathcal{V}_x) turns the quotient manifold

\mathcal{M}/\sim into a *Riemannian submersion* of (\mathcal{M}, g) [37, Section 3.6.2]. This special construction allows for a convenient matrix representation of the gradient [37, Section 3.6.2] and the Hessian [37, Proposition 5.3.3] on the quotient manifold \mathcal{M}/\sim . Below we show the gradient and Hessian computations for the problem (27).

The Riemannian gradient $\text{grad}_{[x]}f$ of f on \mathcal{M}/\sim is uniquely represented by its horizontal lift in \mathcal{M} which has the matrix representation

$$\begin{aligned} &\text{horizontal lift of } \text{grad}_{[x]}f \\ &= \text{grad}_x f = \left(\frac{\partial f}{\partial \mathbf{U}} (\mathbf{V}^T \mathbf{V})^{-1}, \frac{\partial f}{\partial \mathbf{V}} (\mathbf{U}^T \mathbf{U})^{-1} \right), \end{aligned} \quad (32)$$

where $\text{grad}_x f$ is the gradient of f in \mathcal{M} and $\partial f / \partial \mathbf{U}$ and $\partial f / \partial \mathbf{V}$ are the *partial derivatives* of f with respect to \mathbf{U} and \mathbf{V} , respectively.

In addition to the Riemannian gradient computation (32), we also require the directional derivative of the gradient along a search direction. This is captured by a *connection* $\nabla_{\xi_x} \eta_x$, which is the *covariant derivative* of vector field η_x with respect to the vector field ξ_x . The Riemannian connection $\nabla_{\xi_{[x]}} \eta_{[x]}$ on the quotient manifold \mathcal{M}/\sim is uniquely represented in terms of the Riemannian connection $\nabla_{\xi_x} \eta_x$ in the total space \mathcal{M} [37, Proposition 5.3.3] which is

$$\text{horizontal lift of } \nabla_{\xi_{[x]}} \eta_{[x]} = \Pi_x(\nabla_{\xi_x} \eta_x), \quad (33)$$

where $\xi_{[x]}$ and $\eta_{[x]}$ are vector fields in \mathcal{M}/\sim and ξ_x and η_x are their horizontal lifts in \mathcal{M} . Here $\Pi_x(\cdot)$ is the projection operator defined in (31). It now remains to find out the Riemannian connection in the total space \mathcal{M} . We find the matrix expression by invoking the *Koszul* formula [37, Theorem 5.3.1]. After a routine calculation, the final expression is [41]

$$\nabla_{\xi_x} \eta_x = D\eta_x[\xi_x] + (\mathbf{A}_U, \mathbf{A}_V), \text{ where}$$

$$\begin{aligned} \mathbf{A}_U &= \eta_U \text{Sym}(\xi_V^T \mathbf{V})(\mathbf{V}^T \mathbf{V})^{-1} + \xi_U \text{Sym}(\eta_V^T \mathbf{V})(\mathbf{V}^T \mathbf{V})^{-1} \\ &\quad - U \text{Sym}(\eta_V^T \xi_V)(\mathbf{V}^T \mathbf{V})^{-1} \\ \mathbf{A}_V &= \eta_V \text{Sym}(\xi_U^T \mathbf{U})(\mathbf{U}^T \mathbf{U})^{-1} + \xi_V \text{Sym}(\eta_U^T \mathbf{U})(\mathbf{U}^T \mathbf{U})^{-1} \\ &\quad - V \text{Sym}(\eta_U^T \xi_U)(\mathbf{U}^T \mathbf{U})^{-1} \end{aligned} \quad (34)$$

and $D\xi[\eta]$ is the Euclidean directional derivative $D\xi[\eta] := \lim_{t \rightarrow 0} (\xi_{x+t\eta_x} - \xi_x)/t$. $\text{Sym}(\cdot)$ extracts the symmetric part of a square matrix, i.e., $\text{Sym}(\mathbf{Z}) = (\mathbf{Z} + \mathbf{Z}^T)/2$.

The directional derivative of the Riemannian gradient in the direction $\xi_{[x]}$ is given by the *Riemannian Hessian operator* $\text{Hess}_{[x]}f[\xi_{[x]}]$ which is now directly defined in terms of the Riemannian connection ∇ . Based on (33) and (34), the horizontal lift of the Riemannian Hessian in \mathcal{M}/\sim has the matrix expression:

$$\text{horizontal lift of } \text{Hess}_{[x]}f[\xi_{[x]}] = \Pi_x(\nabla_{\xi_x} \text{grad}_x f), \quad (35)$$

where $\xi_{[x]} \in T_{[x]}(\mathcal{M}/\sim)$ and its horizontal lift $\xi_x \in \mathcal{H}_x$. $\Pi_x(\cdot)$ is the projection operator defined in (31).

B. Retraction

An iterative optimization algorithm involves computing a search direction (e.g., negative gradient) and then “moving in that direction”. The default option on a Riemannian manifold

is to move along geodesics, leading to the definition of the *exponential map*. Because the calculation of the exponential map can be computationally demanding, it is customary in the context of manifold optimization to relax the constraint of moving along geodesics. To this end, we define *retraction* $R_x : \mathcal{H}_x \rightarrow \mathcal{M} : \xi_x \mapsto R_x(\xi_x)$ [37, Definition 4.1.1]. A natural update on the manifold \mathcal{M} is, therefore, based on the update formula $x_+ = R_x(\xi_x)$, i.e., defined as

$$\begin{aligned} R_U(\xi_U) &= \mathbf{U} + \xi_U \\ R_V(\xi_V) &= \mathbf{V} + \xi_V, \end{aligned} \quad (36)$$

where $\xi_x = (\xi_U, \xi_V) \in \mathcal{H}_x$ is a search direction and $x_+ \in \mathcal{M}$. It translates into the update $[x_+] = [R_x(\xi_x)]$ on \mathcal{M}/\sim .

C. Riemannian Trust-Region Algorithm

Analogous to trust-region algorithms in the Euclidean space [44, Chapter 4], trust-region algorithms on a Riemannian quotient manifold with guaranteed superlinear rate convergence and global convergence have been proposed in [37, Chapter 7]. At each iteration we solve the *trust-region sub-problem* on the quotient manifold \mathcal{M}/\sim . The trust-region sub-problem is formulated as the minimization of the *locally-quadratic* model of the objective function, say $f : \mathcal{M} \rightarrow \mathbb{R}$ at $x \in \mathcal{M}$,

$$\begin{aligned} &\underset{\xi_x \in \mathcal{H}_x}{\text{minimize}} && g_x(\xi_x, \text{grad}_x f) + \frac{1}{2} g_x(\xi_x, \text{Hess}_x f[\xi_x]) \\ &\text{subject to} && g_x(\xi_x, \xi_x) \leq \Delta^2, \end{aligned} \quad (37)$$

where Δ is the trust-region radius, g_x is the Riemannian metric (29), and $\text{grad}_x f$ and $\text{Hess}_x f$ are the Riemannian gradient and Riemannian Hessian operations defined in (32) and (35), respectively.

Solving the above trust-region sub-problem (37) leads to a direction ξ_x that minimizes the quadratic model. Depending on whether the decrease of the cost function is sufficient or not, the potential iterate is accepted or rejected. The concrete matrix characterizations of Riemannian gradient (32), Riemannian Hessian (35), projection operator (31), and retraction (36) allow to use an *off-the-shelf* trust-region implementation on manifolds, e.g., in Manopt [40], which implements [37, Algorithm 1] that solves the trust-region sub-problem inexactly at every iteration.

The Riemannian trust-region algorithm is *globally convergent*, i.e., it converges to a critical point starting from any random initialization. The rate of convergence analysis of the algorithm is in [37, Chapter 7]. Theoretically, the algorithm converges to a critical point, but often in practice the convergence is observed to a local minimum. Under certain regularity conditions, the trust-region algorithm shows a *superlinear* rate of convergence locally near a critical point. The recent work [45] also establishes *worst-case* global rates (i.e., number of iterations required to obtain a fixed accuracy) of convergence over manifolds. In practice, however, we observe better rates.

D. Computational Complexity

The numerical complexity of the algorithm in Algorithm 1 depends fixed-rank Riemannian optimization algorithm for solving (20), (22), and (23) and sorting the diagonal entries of rank- r matrix. The sorting operation depends linearly with K

(and logarithmic factors of K). The computational cost of the Riemannian algorithm depends on i) the computational cost of the computing the partial derivatives of the objective functions in (20), (22), and (23) and ii) the manifold-related operations. The computational cost of the manifold-related ingredients are shown below.

- 1) Computation of partial derivatives of the objective functions in (20), (22), and (23) with respect to \mathbf{U} and \mathbf{V} : $O(|\mathcal{V}|r)$.
- 2) Computation of Riemannian gradient with the formula (32): $O(Kr^2 + r^3)$.
- 3) Computation of the projection operator (31): $O(Kr^2 + r^3)$.
- 4) Computation of retraction $R_{\bar{x}}$ in (36): $O(Kr)$.
- 5) Computation of Riemannian Hessian with the formulas (33), (34), and (35): $O(r^3 + Kr^2)$.

It is clear that all the manifold-related operations are of linear complexity in K and cubic in r . Overall, the cost per iteration of the proposed algorithm in Algorithm 1 is *linear* with $|\mathcal{V}|$.

VI. SIMULATION RESULTS

In this section, we simulate our proposed Riemannian optimization algorithm for user admission control in topological interference management. All the Riemannian algorithms for the rank-constrained optimization problems (20), (22) and (23) are implemented based on the manifold optimization toolbox Manopt [40]. As the second-order Riemannian trust-region method is robust to the initial points, all the Riemannian algorithms are initialized randomly and terminated when either the norm of the Riemannian gradient is below 10^{-6} or the number of iterations exceeds 500. We set $\|\mathcal{P}_{\Omega}(\mathbf{X}(\mathcal{S}^{[m]})) - \mathbf{I}_{|\mathcal{S}^{[m]}|}\|_F / \sqrt{K} = 10^{-3}$ in (22) to check if the affine constraint $\mathcal{P}_{\Omega}(\mathbf{X}(\mathcal{S}^{[m]})) = \mathbf{I}_{|\mathcal{S}^{[m]}|}$ is satisfied.

The proposed algorithm is compared to the following approaches:

- **Exhaustive search:** This is achieved by solving a sequence of problem $\mathcal{F}(\mathcal{S}^{[m]})$ using (22) via exhaustively searching over the set $\mathcal{S}^{[m]} \subseteq \{1, \dots, K\}$. We use the Riemannian trust-region algorithm in Section V to solve (22). However, the complexity of exhaustive search grows exponentially in the number of users K .
- **Orthogonal scheduling:** In the conventional orthogonal schemes such as TDMA/FDMA, one can only achieve a symmetric DoF allocation $1/K$ per user [13]. In this case, given the symmetric DoF allocation $1/r$, the number of admitted users equals r .

A. Admitted Users versus Achievable DoFs

Consider a 8-user partially connected interference channel with $|\mathcal{V}| = 45$ interference channel links. The sets of the connected interference links are generated uniformly at random. The proposed three-stage Riemannian algorithm based user admission approach is compared with the exhaustive search and the orthogonal scheduling approaches. We set $\lambda = 0.5$,

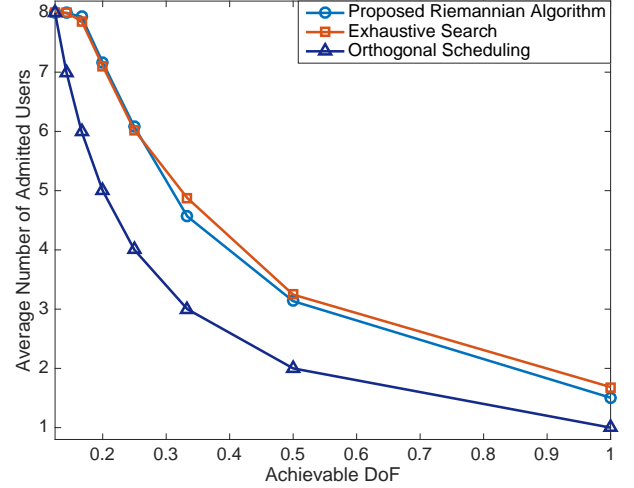


Fig. 5. Average number of admitted users versus the achievable DoFs with different algorithms.

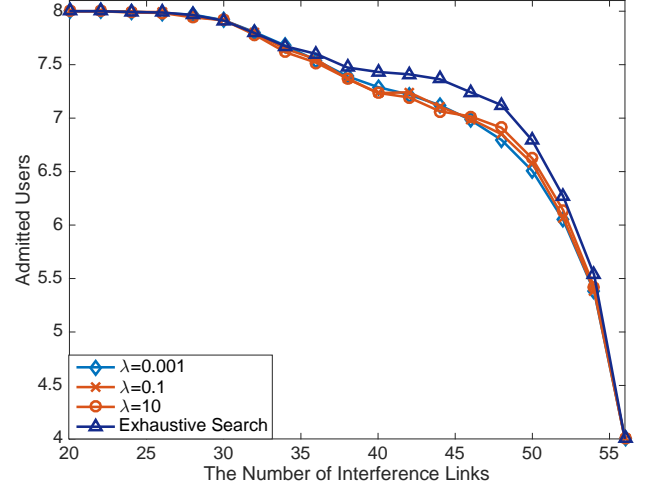


Fig. 6. Different weighting parameters λ in (20).

$\rho = 0.01$ and $\epsilon = 0.01$ in the sparse inducing optimization problem (20). Fig. 5 demonstrates the average number of admitted users with different symmetric DoF allocations. Each point in the simulations is averaged over 500 randomly generated network topology realizations \mathcal{V} . From Fig. 5, we can see that the proposed three-stage Riemannian algorithm achieves near-optimal performance compared with exhaustive search and significantly outperforms the conventional orthogonal scheduling scheme.

B. Different Values of the Weighting Parameter λ

Consider a 8-user partially connected interference channel. The sets of the connected interference links are generated uniformly at random. We set $\rho = 0.01$, $\epsilon = 0.01$ and $r = 4$ in the sparse inducing optimization problem (20). Fig. 6 shows the average number of admitted users with different values of the weighting parameter λ in the regularized smoothed

ℓ_1 -norm in (20). Each point in the simulations is averaged over 500 randomly generated network topology realizations \mathcal{V} . Fig. 6 demonstrates that parameter λ does not affect the induced sparsity pattern in $\text{diag}(\mathbf{X})$, thereby yielding almost the same number of admitted users. The reason is that the role of the weighting parameter λ in (20) only serves to upper bound the objective function. This figure further indicates that the proposed Riemannian algorithm achieves near optimal performance with the exhaustive search approach and outperforms the orthogonal scheduling scheme with $r = 4$, i.e., the number of admitted users is 4.

VII. CONCLUSIONS AND DISCUSSIONS

This paper presented a sparse and low-rank optimization framework for user admission control in topological interference management. A Riemannian optimization framework was further developed to solve the non-convex rank-constrained ℓ_0 -norm maximization problem, supported by a novel regularized smoothed ℓ_1 -norm sparsity inducing minimization approach. In particular, by exploiting the quotient manifold of fixed-rank matrices, we presented a Riemannian trust-region algorithm to find good solutions to the non-convex sparse and low-rank optimization problem. Simulation results illustrated the effectiveness and near-optimal performance of the proposed algorithms.

Several future directions of interest are as follows:

- It is desirable but challenging to theoretically establish the fundamental tradeoffs between the sparsity and low-rankness in the sparse and low-rank model \mathcal{P} .
- It is particularly interesting and also important to apply the sparse and low-rank framework to more important problems including the index coding problem [46], caching networks [47], [48], and distributed computing systems [49], thereby investigating the fundamental limits of communication, computation and storage.
- It is also interesting to apply the Riemannian optimization technique to other important network optimization problems, e.g., blind deconvolution for massive connectivity in Internet-of-Things (IoT) [50] and the hybrid precoding in millimeter wave systems [51].

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